Systèmes entrée-sortie non linéaires et applications en audio-acoustique

Séries de Volterra

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Ecole Thématique "Théorie du Contrôle en Mécanique"
2019

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Outline

Exercice 1: mechanical damped Duffing oscillator

Consider the Duffing oscillator excited by a force f, governed by

$$(S): \quad m\ddot{x} + a\dot{x} + k_1 x + k_3 x^3 = f$$

(zero initial conditions)

We are interested in $f \rightarrow \boxed{\{H_n\}} \rightarrow x$

Questions

1. Draw the block-diagram of the canceling system of (S).
2. Derive the equations satisfied by the transfer kernels $\{H_n\}$ for all $n \geq 1$.
3. Derive the transfer kernels up to order 5.
4. Propose a realization for these orders. *(composed of linear systems and static nonlinear functions)*

Exercice 2: formal inverse kernels (open-loop control)

Consider a system $u \rightarrow \boxed{\{F_n\}} \rightarrow v$.

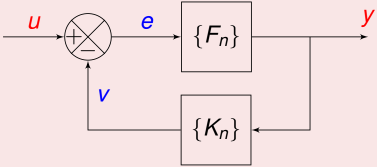
Questions

1. Derive the equations satisfied by the transfer kernels $v \rightarrow \boxed{\{G_n\}} \rightarrow u$.

   (use the abstract relation $w \rightarrow \boxed{\{A_n\}} \rightarrow \boxed{\{B_n\}} \rightarrow w$ to express A w.r.t B or vice-versa)

2. Derive the transfer kernels G_n up to order 3.

Exercice 3: formal kernels of a closed-loop system



Signals: u=reference, e=measured error, y=system output, v=measured output
Systems: $\{F_n\}$=plant *(or controller+plant)*, $\{K_n\}$=sensor

Questions

1. Draw the canceling system associated with $u \to \boxed{\{H_n\}} \to y$
2. Derive the equations satisfied by the transfer kernels $\{H_n\}$ for all $n \geq 1$.
3. Derive $\{H_n\}$ up to order 3

Rk: *you can introduce the kernels* $\{G_n\}$ *of* $u \to \boxed{\{H_n\}} \to \boxed{\{K_n\}} \to v$ *and first express* $\{H_n\}$ *w.r.t.* $\{F_n\}$ *and* $\{G_n\}$.

Outline

Simulation of the weakly nonlinear propagation in a straight pipe:

Application to a real-time brassy audio effect

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MED'08, June 2008, AJACCIO

Purpose (audio effects and sound synthesis)

- Simulate the realistic propagation of a progressive plane wave in a pipe

- Include the **nonlinearity** responsible for the **brightness of « brass sounds »** at **fortissimo** nuances (|p|<160 dB spl)
- Low-cost input/output relation

  Choice: Volterra series

Outline

1. Nonlinear acoustic model
(planar progressive wave)

- [Mak97] adimensional version:

For x>0, t>0, $$\partial_x p + \partial_t p + A(p) = \frac{\beta}{2} \partial_t p^2$$

Boundary Cnd.: $$p(x=0,t) = p_0(t)$$ (input)

- Damping models A(p):

Simplest: $A_0(p) = \alpha_0 \, p$

Realistic (brass instr., **[MJ00]**): $A_1(p) = \alpha_1 \, \partial^{1/2} p$

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- Fractional derivative (e.g.[Mat98]):

$\partial_t^{1/2} \Longrightarrow \times\sqrt{s}$ (Laplace domain)

with $s = \rho e^{i\theta} \mapsto \sqrt{s} = \sqrt{\rho} e^{i\theta/2}$, $\rho \geq 0$, $\theta \in [-\pi, \pi[$

3. Volterra kernels of the acoustic pb.

How to derive adapted Volterra kernels ?

see e.g. **[Hélie,Hasler 2003]**

Kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$ and cancelling system

- For x>0, t>0, $\partial_x p + \partial_t p + \alpha_k \partial_t^{k/2} p = \frac{\beta}{2}\partial_t p^2$

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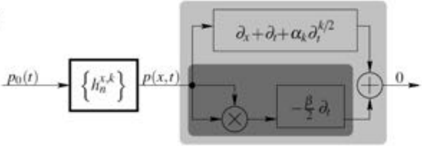

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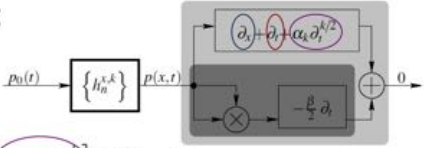

- From laws:

$$\partial_x H_n^{x,k}(s_{1:n}) + \left[\widehat{s_{1:n}} + \alpha_k \left(\widehat{s_{1:n}}\right)^{\frac{k}{2}}\right] H_n^{x,k}(s_{1:n})$$

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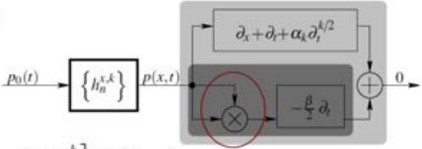

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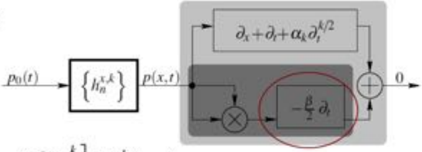

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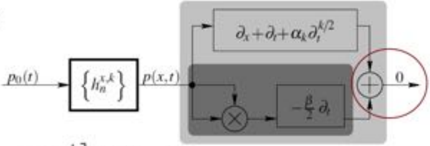

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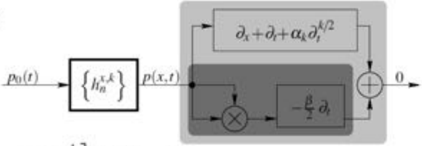

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Linear ODEs

Boundary cnd. and solution

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$$
\begin{aligned}
G_1^{x,k}(s_{1:n}) &= \mathrm{e}^{-\alpha_k s_1^{\frac{k}{2}}} \\
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- First kernels (k=0)

$$G_1^{x,0}(s_1) = \mathrm{e}^{-\alpha_0 x}$$

$$G_2^{x,0}(s_{1:2}) = \frac{\beta\,\widehat{s_{1:2}}}{2\alpha_0}(1-\mathrm{e}^{-\alpha_0 x})$$

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- First kernels (k=1)

$$G_1^{x,1}(s_1) = \mathrm{e}^{-\alpha_1 x\sqrt{s_1}},$$

$$G_2^{x,1}(s_{1:2}) = \frac{\beta\widehat{s_{1:2}}}{2\alpha_1} \frac{\mathrm{e}^{-\alpha_1 x\sqrt{s_1+s_2}} - \mathrm{e}^{-\alpha_1 x(\sqrt{s_1}+\sqrt{s_2})}}{-\sqrt{s_1+s_2}+\sqrt{s_1}+\sqrt{s_2}}$$

Deriving simple realizable structures

How to realize first kernels without multi-convolutions ?

Deriving simple realizable structures

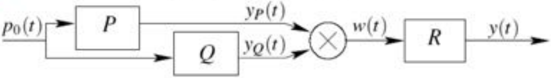
How to realize first kernels without multi-convolutions ?

n=1: linear filter (mono-conv.)

What about n=2 ?

Elementary 2nd order system

- Elementary system (P,Q,R: transfer fct):



$$K_2(s_{1:2}) = P(s_1)\,Q(s_2)\,R(\widehat{s_{1:2}}) \quad \text{if } n = 2,$$
$$K_n(s_{1:n}) = 0 \quad \text{otherwise.}$$

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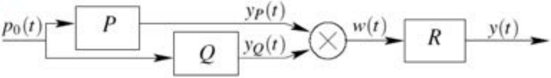

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- For k=0, $G_2^{x,0}(s_{1:2}) = \dfrac{\beta\,\widehat{s_{1:2}}}{2\alpha_0}(1-e^{-\alpha_0 x})$

$$
\begin{aligned}
P(s)=Q(s) &= 1, \text{ (identity systems)}\\
R(s) &= \frac{\beta\left(1-e^{-\alpha_0 x}\right)}{2\alpha_0}\,s,
\end{aligned}
$$

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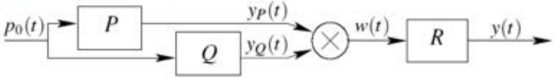

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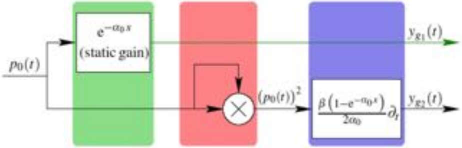

Realistic case: k=1

- No straightforward identification:

$$G_2^{x,1}(s_{1:2}) = \frac{\beta \widehat{s_{1:2}}}{2\alpha_1} \frac{e^{-\alpha_1 x\sqrt{s_1+s_2}} - e^{-\alpha_1 x(\sqrt{s_1}+\sqrt{s_2})}}{-\sqrt{s_1+s_2}+\sqrt{s_1}+\sqrt{s_2}}$$

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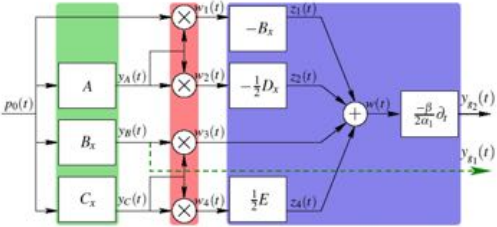
- Perfect squares & sum of elementary syst.:

$$\left[\frac{\sqrt{s_1+s_2}+\sqrt{s_1}+\sqrt{s_2}}{\sqrt{s_1+s_2}+\sqrt{s_1}+\sqrt{s_2}} \cdot G_2^{x,k=1}(s_{1:2})\right]$$
$$= \Big[A(s_1)\mathbf{1}(s_2)B_x(\widehat{s_{1:2}}) + \mathbf{1}(s_1)A(s_2)B_x(\widehat{s_{1:2}})$$
$$+A(s_1)A(s_2)D_x(\widehat{s_{1:2}})$$
$$-B_x(s_1)C_x(s_2)\mathbf{1}(\widehat{s_{1:2}}) - C_x(s_1)B_x(s_2)\mathbf{1}(\widehat{s_{1:2}})$$
$$-C_x(s_1)C_x(s_2)E(\widehat{s_{1:2}})\Big]\frac{\beta}{4\alpha_1}\widehat{s_{1:2}},$$

Realistic case: 2nd order realization



Structure composed of sums, products and linear filters with (irrational) transfer functions:

$A(s) = 1/\sqrt{s}$ $\quad B_x(s) = G_1^{x,1}(s) = e^{-\alpha_1 x\sqrt{s}}$

$C_x(s) = e^{-\alpha_1 x\sqrt{s}}/\sqrt{s}$ $\quad D(s) = \sqrt{s}e^{-\alpha_1 x\sqrt{s}}$ $\quad E(s) = \sqrt{s}$

Simulation and results

How to obtain accurate digital versions ?

Accurate digital versions

- Irrational transfer functions A,B,C,D,E:

  [fft,product,ifft]: off line solution

  [Hélie,Matignon06]: finite order approx. based on integral/diffusive representations

Accurate digital versions

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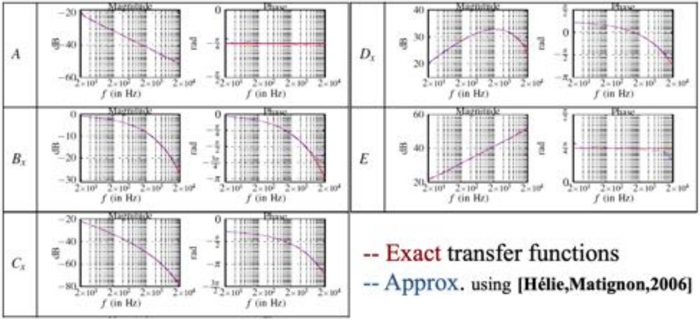
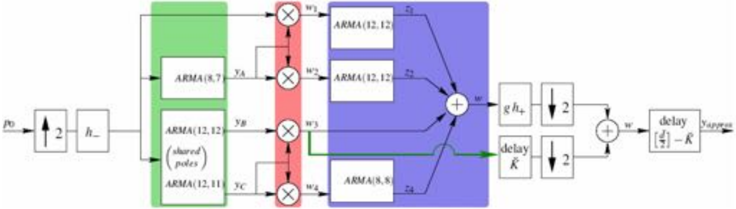
- <u>Product of 2 signals (time domain):</u>
  frequency range is [0,f]+[0,f] = [0,2f]
  oversampling factor = 2 ⇒ no aliasing

Bode diagrams of A,B,C,D,E for typical pipes

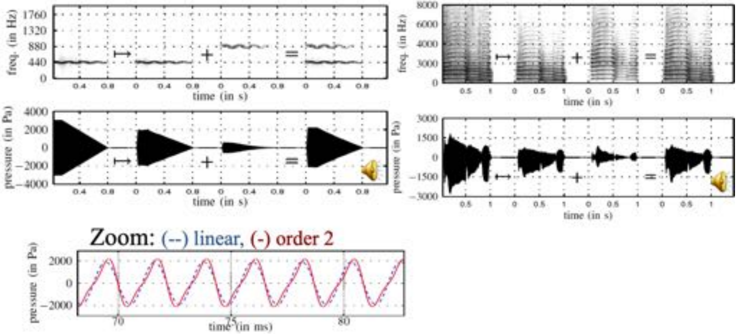


Digital 2nd order realization



Results for a typical trumpet pipe

- Ex.: 1.sinusoid with vibrato / 2.Chet Baker



Conclusion and perspectives

- « **Physically-based** » **brassy effect** for non stationary signals
- **Real-time version** (Plug-In VST, **R. Muller**)
- **Other results for sound applications**: String (**D. Roze**) , NL audio circuits, etc
- **No result about the convergence** for PDEs (in progress for ODEs, **B. Laroche**)
- Method to **accelerate the convergence** under study

Outline

Correction de l'effet Doppler d'une membrane de haut-parleur

Tristan Lebrun[1], Thomas Hélie[1]

[1]Equipe S3AM : Systèmes et Signaux Sonores : Audio/Acoustique, InstruMents
Laboratoire STMS
UMR 9912 IRCAM-CNRS-Sorbonne Université

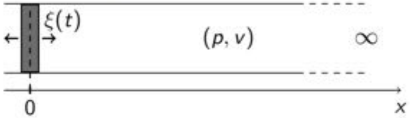
27 avril 2018

Description du phénomène

- **Contexte :** thèse sur la correction des distorsions d'un haut-parleur électrodynamique
- **Phénomène :** Le mouvement de la membrane introduit un temps de propagation variable
- **Conséquence :** Distorsion de l'onde sonore résultante

Modèle et hypothèses

- Modèle repris de Butterweck : piston plan dans un tube semi-infini



Hypothèses

(H1) ondes planes

(H2) pas d'onde retour

(H3) pas d'onde de choc

(H4) $\xi(t)$ décrit la position du piston

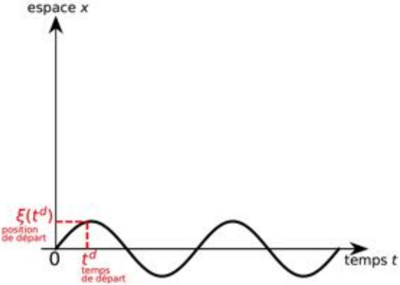
Forme de la solution

$$\mathbf{v}(\mathbf{x}, \mathbf{t}) = \mathbf{v}^+(\mathbf{t} - \mathbf{x}/\mathbf{c}_0)$$

tel que

$$\mathbf{v}(\xi(\mathbf{t}), \mathbf{t}) = \xi'(\mathbf{t})$$

Méthode des caractéristiques



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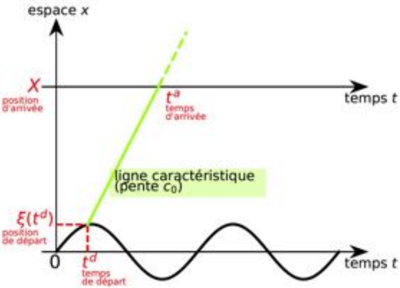

- Relation entre les vitesses :

$$v(X, t^a) = v(\xi(t^d), t^d) = \xi'(t^d)$$

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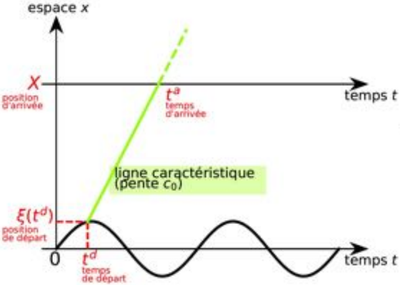


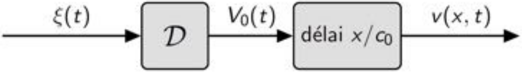
- Relation entre les vitesses :

$$\begin{aligned} v(X, t^a) &= v(\xi(t^d), t^d) \\ &= \xi'(t^d) \end{aligned}$$

- Relation entre les temps :

$$t^a = t^d + \frac{X - \xi(t^d)}{c_0}$$

Solution générale



La source eulérienne équivalente en $x = 0$ s'exprime

$$V_0(t) = \xi'\Big(t + \epsilon(t)\Big)$$

avec

$$\underbrace{\epsilon(t) = \xi\Big(t + \epsilon(t)\Big)/c_0}_{\text{équation implicite à résoudre}}$$

Résolution de $\epsilon(t) = \xi\Big(t + \epsilon(t)\Big)/c_0$

Méthode à perturbations régulières

$$\alpha u(t) \longrightarrow \boxed{\mathcal{S}} \longrightarrow y(t) = \alpha y_1(t) + \alpha^2 y_2(t) + \alpha^3 y_3(t) + \ldots$$

$$\sum_{n=1}^{\infty} \alpha^n \epsilon_n(t) = \alpha \xi\Big(t + \sum_{m=1}^{\infty} \alpha^m \epsilon_m(t)\Big)/c_0$$

1. Méthode à perturbations régulières

Résolution de $\epsilon(t) = \xi\Big(t + \epsilon(t)\Big)/c_0$

Méthode à perturbations régulières

$$\alpha u(t) \longrightarrow \boxed{\mathcal{S}} \longrightarrow y(t) = \alpha y_1(t) + \alpha^2 y_2(t) + \alpha^3 y_3(t) + \ldots$$

$$\sum_{n=1}^{\infty} \alpha^n \epsilon_n(t) = \alpha \sum_{p=0}^{\infty} \frac{\xi^{(p)}(t)}{p!} \Big(\sum_{m=1}^{\infty} \alpha^m \epsilon_m(t) \Big)^p / c_0$$

1. Méthode à perturbations régulières
2. Développement en séries de Taylor

Résolution de $\epsilon(t) = \xi\Big(t + \epsilon(t)\Big)/c_0$

Méthode à perturbations régulières

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1. Méthode à perturbations régulières
2. Développement en séries de Taylor

Regroupement des α^n **:** $\epsilon_n(t) = f_\xi\Big(\epsilon_1, \ldots, \epsilon_{n-1}(t)\Big)$

$$\epsilon(t) = \frac{\xi(t)}{c_0} + \frac{\xi(t)\xi'(t)}{c_0^2} + \frac{1}{c_0^3}\Big(\frac{\xi(t)^2\xi''(t)}{2} + \xi(t)\xi'(t)^2\Big) + \ldots$$

Source eulérienne $V_0(t) = \xi'\Big(t + \epsilon(t)\Big)$

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Source eulérienne $V_0(t) = \xi'\Big(t + \epsilon(t)\Big)$

$$V_0(t) = \xi'\big(t + \sum_{n=1}^{\infty} \epsilon_n(t)\big)$$

1. Injection de $\epsilon(t) = \sum_{n=1}^{\infty} \epsilon_n(t)$

Source eulérienne $V_0(t) = \xi'\Big(t + \epsilon(t)\Big)$

$$V_0(t) = \sum_{n=0}^{\infty} \frac{\xi^{n+1}(t)}{n!} \Big(\sum_{n=1}^{\infty} \epsilon_n(t)\Big)^n$$

1. Injection de $\epsilon(t) = \sum_{n=1}^{\infty} \epsilon_n(t)$
2. Développement en séries de Taylor

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1. Injection de $\epsilon(t) = \sum_{n=1}^{\infty} \epsilon_n(t)$
2. Développement en séries de Taylor

   Calcul par composition de séries entières :

$$V_0(t) = \underbrace{\xi'(t)}_{\text{linéaire}} + \underbrace{\frac{\xi(t)\xi''(t)}{c_0} + \frac{1}{c_0^2}\Big(\xi''(t)\xi'(t)\xi(t) + \frac{\xi(t)^2\xi'''(t)}{2}\Big) + ...}_{\text{non linéaire}}$$

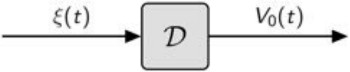
Remarques

- Rayon de convergence de la série entière ?
- Somme de polynômes en $(\xi, ..., \xi^{(n)})$ homogènes d'ordre n

Reformulation en séries de Volterra

- $V_0(t) = \sum_{n=0}^{\infty} v_n(t)$
- chaque $v_n(t)$ est un polynôme de degré homogène n

$$
\begin{aligned}
v_1(t) &= \xi'(t) : \text{terme linéaire} \\
v_2(t) &= \frac{\xi''(t)\xi(t)}{c_0} : \text{terme quadratique}
\end{aligned}
$$



Noyaux de transfert $\mathcal{D}_n$ du système $\mathcal{D}$:

$$
\begin{aligned}
D_1(s_1) &= s_1 : \text{terme linéaire} \\
D_2(s_1, s_2) &= \frac{s_1^2}{c_0} : \text{terme quadratique}
\end{aligned}
$$

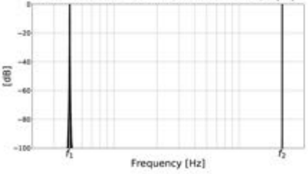
L'effet Doppler est-il négligeable pour les HP?

Paramètres de simulation

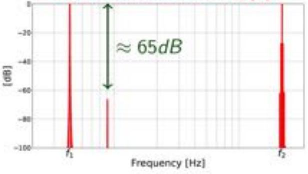
- Excitation bi-harmonique de fréquence $f_1 = 40\text{Hz}$, $f_2 = 2\text{kHz}$.
- Amplitude 1 m/s (déplacement $< 4\text{mm}$)

$$\xi(t) \longrightarrow \boxed{\mathcal{D}} \longrightarrow V_0(t)$$

Vitesse de la membrane $\xi'(t)$



Source eulerienne $V_0(t)$



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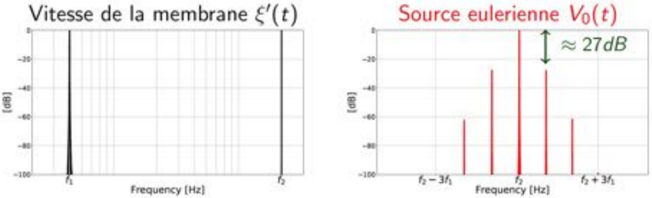
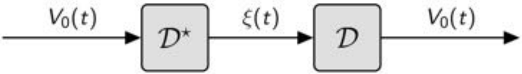


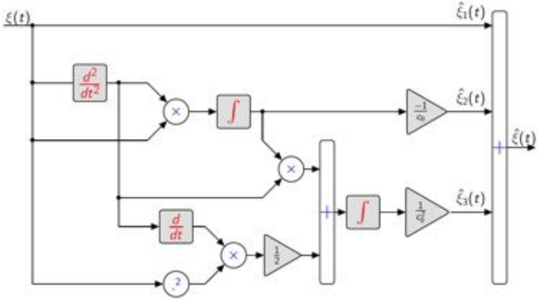
Calcul des noyaux post-inverses



- **Méthode d'inversion des noyaux**
- On calcule les noyaux de transfert du système $\mathcal{D}^\star$

$$\begin{aligned} D_1^\star(s_1) &= 1/s_1 \text{ : terme linéaire de correction} \\ D_2^\star(s_1, s_2) &= -\frac{s_1}{c_0 s_2(s_1+s_2)} \text{ : terme quadratique de correction} \\ D_3^\star(s_1, s_2, s_3) &= \ldots \end{aligned}$$

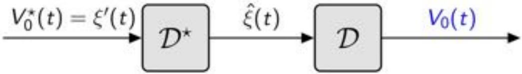
Synthèse du correcteur à l'ordre 3

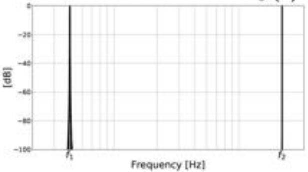


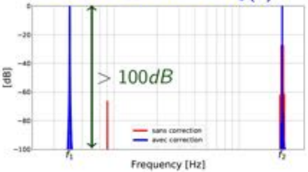
Algorithme composé de...

- dérivateurs et intégrateurs
- produits et sommes entre signaux

Evaluation numérique







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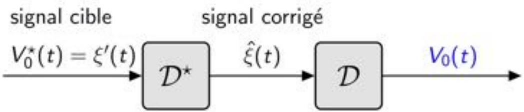

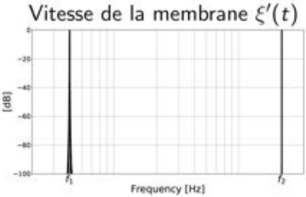

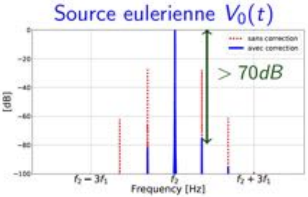

Conclusion

Bilan

- Elaboration d'un solveur pour le problème direct par troncature de la série formelle décrivant le phénomène
- Taux d'intermodulation non négligeable pour des signaux larges bandes dans les HP
- Construction d'un correcteur restaurant une dynamique de 70 dB

Perspectives

- Amélioration du modèle (convection)
- Propriétés de convergence de $V_0(t)$
- Essais expérimentaux du correcteur sur large membrane

Outline

Volterra series for real-time simulations of weakly nonlinear analog audio devices: Application to the Moog Ladder Filter

Thomas Hélie

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Équipe Analyse/Synthèse
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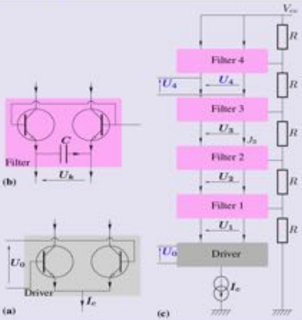
DAFx-06

The Moog ladder filter: dimensionless equations

Circuit



Filter k ($k = 1, 2, 3, 4$)

$$\frac{1}{\omega_C}\frac{\mathrm{d}u_k}{\mathrm{d}t} + \tanh u_k = \tanh u_{k-1}$$

with $u_k = \frac{U_k}{2V_T}$, $\omega_C = \frac{I_c}{4CV_T}$,

Loop between U_4 and U_0

$u_0 = u_{in} - 4r\, u_4$ with $r \in [0,1]$

WNL properties are satisfied:

- tanh admits a power series
- ONLY 1 equilibrium point

Outline

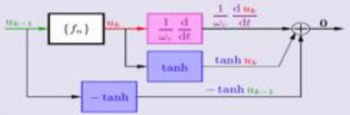
1. Introduction
   - Weakly nonlinear analog circuits
2. Volterra series
   - Definition and properties
3. **Case of the Moog ladder filter**
   - Derivation of the Volterra kernels
4. Simulation
   - Realizable structures and Results
5. Conclusion

Kernels $\{f_n\}$ of a Moog single stage $u_{k-1} \rightarrow \{f_n\} \rightarrow u_k$

Equation

$$\frac{1}{\omega_c}\frac{\mathrm{d}u_k}{\mathrm{d}t} + \tanh u_k = \tanh u_{k-1}$$

Block diagr.: *canceling system*



Elementary blocks → Volterra kernels in Lapl. domain

$\frac{1}{\omega_c}\frac{\mathrm{d}}{\mathrm{d}t}$ → $Q_1(s_1) = \frac{s_1}{\omega_c}$, $Q_n = 0$ if $n \geq 2$

$\tanh(x) = \sum_{p=1}^{+\infty} T_{2p-1} x^{2p-1}$ → constant kernels $\{T_n\}$

with $T_1 = 1$, $T_3 = -1/3$, $T_{2p-1} = (-1)^{p-1} 2(2^{2p}-1)B_{2p}/(2p)!$
(B_n: Bernoulli numbers)

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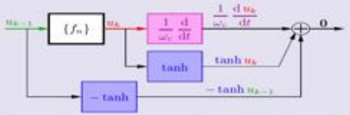

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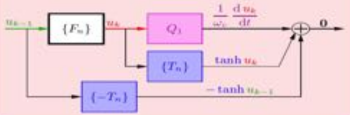

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Kernels $\{f_n\}$ of a Moog single stage $\xrightarrow{u_{k-1}} \boxed{\{f_n\}} \xrightarrow{u_k}$

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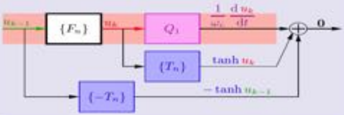


Kernels of order *n* of the *canceling system*

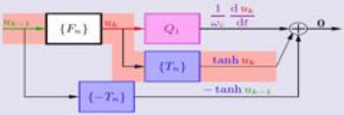
$$F_n(s_{1,n})Q_1(s_1 + \dots + s_n)$$

Kernels $\{f_n\}$ of a Moog single stage

Equation

$$\frac{1}{\omega_c}\frac{\mathrm{d}u_k}{\mathrm{d}t}+\tanh u_k=\tanh u_{k-1}$$

Block diagr.: *canceling system*



Kernels of order *n* of the *canceling system*

$$F_n(s_{1,n})Q_1(s_1+\dots+s_n)$$

$$\sum_{p=1}^{n}\sum_{(i_1,\dots,i_p)\in\mathbb{I}_n^p} F_{i_1}(s_{1,i_1})\dots F_{i_p}(s_{i_1+\dots+i_{p-1}+1,n})T_p$$

Kernels $\{f_n\}$ of a Moog single stage $u_{k-1} \rightarrow \{f_n\} \rightarrow u_k$

Equation

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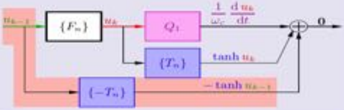


Kernels of order *n* of the *canceling system*

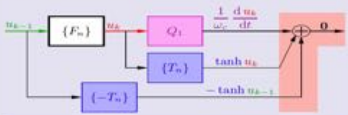
$$F_n(s_{1,n})Q_1(s_1 + \ldots + s_n)$$
$$\sum_{p=1}^{n} \sum_{(i_1,\ldots,i_p)\in\mathbb{I}_n^p} F_{i_1}(s_{1,i_1}) \ldots F_{i_p}(s_{i_1+\ldots+i_{p-1}+1,n}) T_p$$
$$-T_n$$

Kernels $\{f_n\}$ of a Moog single stage $\xrightarrow{u_{k-1}} \boxed{\{f_n\}} \xrightarrow{u_k}$

Equation

$$\frac{1}{\omega_c}\frac{\mathrm{d}u_k}{\mathrm{d}t} + \tanh u_k = \tanh u_{k-1}$$

Block diagr.: *canceling system*



Kernels of order n of the *canceling system*

$$
\begin{aligned}
& F_n(s_{1,n}) Q_1(s_1 + \ldots + s_n) \\
+ & \sum_{p=1}^{n} \sum_{(i_1,\ldots,i_p)\in \mathbb{I}_n^p} F_{i_1}(s_{1,i_1}) \ldots F_{i_p}(s_{i_1+\ldots+i_{p-1}+1,n}) T_p \\
- & T_n \qquad = 0
\end{aligned}
$$

Kernels $\{f_n\}$ of a Moog single stage $u_{k-1} \rightarrow \boxed{\{f_n\}} \rightarrow u_k$

General solution : recursive algebraic equations ($n \geq 1$)

$$F_n(s_{1,n}) = \frac{T_n - \displaystyle\sum_{p=2}^{n} T_p \overbrace{\sum_{(i_1,\dots,i_p)\in\mathbb{I}_n^p} F_{i_1}(s_{1,i_1}) \dots F_{i_p}(s_{i_1+\dots+i_{p-1}+1,n})}^{i_1,\dots,i_p<n}}{T_1 + Q_1(s_1 + \dots + s_n)}$$

with $Q_1(s) = s/\omega_c$, $T_1 = 1$, $T_3 = -1/3$, etc.

First kernels ($n = 1, 2, 3$)

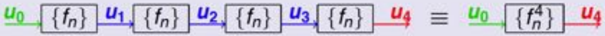
$F_1(s_1) = [T_1 + Q_1(s_1)]^{-1} = \left[1 + \frac{s_1}{\omega_c}\right]^{-1}$ (1^{st} order low-pass filter)

$F_2(s_{1,2}) = 0$

$F_3(s_{1,3}) = [1 - F_1(s_1)F_1(s_2)F_1(s_3)]\, T_3 F_1(s_1 + s_2 + s_3)$

Kernels $\{f_n^4\}$ of a four-stages filter $u_0 \rightarrow \{f_n^4\} \rightarrow u_4$

Block diagram



From the cascade interconnection law: (n=1,2,3)

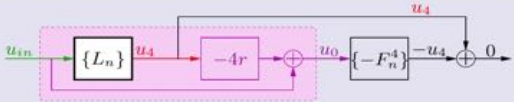
$$F_1^4(s_1) = [F_1(s_1)]^4 = \left[1 + \frac{s_1}{\omega_c}\right]^{-4} \quad \text{(}4^{th}\text{-order low-pass filter)}$$

$$F_2^4(s_{1,2}) = 0$$

$$F_3^4(s_{1,3}) = \sum_{k=0}^{3} [F_1(s_1)]^k [F_1(s_2)]^k [F_1(s_3)]^k F_3(s_{1,3}) [F_1(s_1+s_2+s_3)]^{3-k}$$

Moog ladder filter with a loop $u_{in} \rightarrow \{L_n\} \rightarrow u_4$

Loop $u_0 = u_{in} - 4r\, u_4$: canceling system



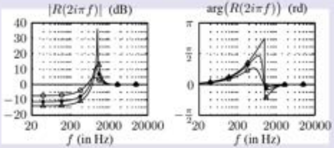
From interconnection laws:

$$L_1(s_1) = R_1(s_1)\, F_1^4(s_1)$$

$$L_2(s_{1,2}) = 0$$

$$L_3(s_{1,3}) = R_1(s_1) R_1(s_2) R_1(s_3) . F_3^4(s_{1,3})\, R_1(s_1+s_2+s_3)$$

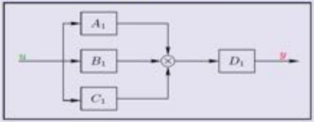
with $R_1(s) = \left[1 + 4r\, F_1^4(s)\right]^{-1}$



Outline

One-stage filter: structure of order 3

Elementary system of order 3

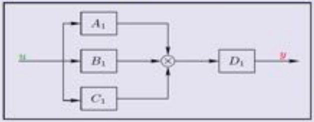


Volterra kernels are 0, except order 3:

$$A_1(s_1)B_1(s_2)C_1(s_3)D_1(s_1+s_2+s_3)$$

One-stage filter: structure of order 3

Elementary system of order 3



Volterra kernels are 0, except order 3:

$A_1(s_1)B_1(s_2)C_1(s_3)D_1(s_1+s_2+s_3)$

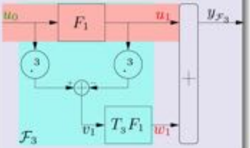
One stage filter kernels

$$F_1(s_1) = \left[1 + \frac{s_1}{\omega_c}\right]^{-1}$$

$$F_2(s_{1,2}) = 0$$

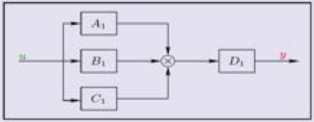
$$F_3(s_{1,3}) = \left[1 - F_1(s_1)F_1(s_2)F_1(s_3)\right] T_3 F_1(s_1+s_2+s_3)$$

IDENTIFICATION



One-stage filter: structure of order 3

Elementary system of order 3



Volterra kernels are 0, except order 3:

$$A_1(s_1)B_1(s_2)C_1(s_3)D_1(s_1+s_2+s_3)$$

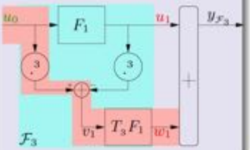
One stage filter kernels

$$F_1(s_1) = \left[1 + \frac{s_1}{\omega_c}\right]^{-1}$$

$$F_2(s_{1,2}) = 0$$

$$F_3(s_{1,3}) = [1 - F_1(s_1)F_1(s_2)F_1(s_3)]\, T_3 F_1(s_1+s_2+s_3)$$

IDENTIFICATION



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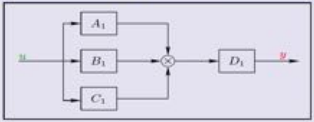

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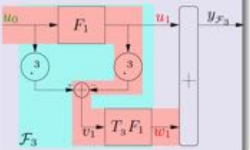


Four-stages filter and loop: strucutre of order 3

Four-stages filter

$$F_1^4(s_1) = [F_1(s_1)]^4$$

$$F_2^4(s_{1,2}) = 0$$

$$F_3^4(s_{1,3}) = \sum_{k=0}^{3} [F_1(s_1)]^k [F_1(s_2)]^k \, . \, [F_1(s_3)]^k F_3(s_{1,3}) [F_1(s_1+s_2+s_3)]^{3-k}$$

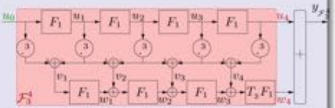
Moog Ladder filter with a loop

$$L_1(s_1) = R_1(s_1)\, F_1^4(s_1)$$

$$L_2(s_{1,2}) = 0$$

$$L_3(s_{1,3}) = R_1(s_1) R_1(s_2) R_1(s_3) \, . F_3^4(s_{1,3})\, R_1(s_1+s_2+s_3)$$

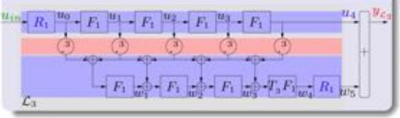
IDENTIFICATION



IDENTIFICATION

Digital implementation and sound example

Structure



Digital implementation

Linear filters: standard methods

Cube powers: reject aliasing
→ oversampling by $k \geq 3$

Example

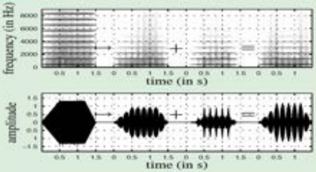
Input: detuned square wave

Filter: $f_c = 1500$Hz, $r = 0.15$

Sampling: $f_s = 3 \times 44100$Hz

$u_{in} \mapsto u_4 + w_5 = y_{\mathcal{L}_3}$

linear — Volterra



Plan

Plan

Plan